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Mathematical Modeling of Transient Heat Conduction and Analysis of Thermal Stresses in a Thin Circular Plate

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Abstract

In this paper, the solution of the problem of transient heat conduction in a thin circular plate subjected to two types of boundary conditions is obtained by employing the integral transform technique in the form of infinite series. It is assumed that the plate is in the plane state of stress and initially the temperature of the plate is kept at zero.

The first type of boundary condition is that in which the upper surface is kept at arbitrary temperature, lower surface is kept at zero temperature and circular edge is insulated. In the literature, the origin of coordinates is taken to be the centre of the lower surface of the plate.

The second type of boundary condition is that in which a linear combination of temperature and its normal derivatives is prescribed on the circular edge as well as on the plane surfaces of the plate.

The true results are given in the form of figure.

Key words

Boundary value problem; Transient heat conduction; Thin circular plate

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1. INTRODUCTION

Roy Choudhuri[1] studied the normal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. The lower face of the plate is kept at zero temperature, while the circular edge is thermally insulated. A more general problem of determining the transient quasi-static thermal deflection of a thin circular plate on upper surface of the plate is studied by Meshram and Deshmukh [2]. The generality of the problem in[2] lies in the fact that the upper face is subjected to arbitrary temperature distribution.

Meshram and Deshmukh[3] considered a thin circular plate of thickness h occupying the space $D : 0 \leq r \leq a, 0 \leq z \leq h$.Initially the temperature of the plate is kept at zero. The upper surface is kept at arbitrary temperature, lower surface is kept at zero and the circular edge is insulated. Let the problem studied in[3] be named as Problem1.Let $T(r, z, t)$ be the temperature of the plate at time t satisfying the equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{1.1}$$

subject to the initial condition

$$[T(r, z, t)]_{t=0} = 0 \tag{1.2}$$

and the boundary conditions

$$[T(r, z, t)]_{z=0} = 0 \tag{1.3}$$

$$[T(r, z, t)]_{z=h} = f(r, t) \tag{1.4}$$

$$\left[\frac{\partial T}{\partial r}\right]_{r=a} = 0 \tag{1.5}$$

where k is the thermal diffusivity of the material of the plate. The heat conduction problem in[2] is the same as the above except that the origin is shifted to the centre of the middle plane of the plate.

Guided by a procedure outlined by Ölcer[5], the so called pseudo-steady temperature distribution function is introduced and an alternative solution is obtained in[3] for Problem 1. The alternative solution in[3] is erroneous and hence different from those obtained by using integral transform methods. Also, since the erroneous alternative solutions are simpler, they are taken to be the solutions of problems under consideration. The correct solutions obtained by using integral transform methods are either used for comparison[3].

In the present paper we find the alternative solution correctly for Problem 1, and show that the so called alternative solution is the same as the one obtained by using integral transform methods.

2. SOLUTION OF THE PROBLEM 1

For function $T(r, z, t)$ let $\hat{T}(\xi_n, z, t)$ denote the finite Hankel transform with respect to r and let $\bar{T}(r, \lambda_m, t)$ denote the finite Fourier sine transform with respect to z then

$$\bar{\hat{T}}(\xi_n, \lambda_m, t) = \int_0^a r J_0(\xi_n r) dr \int_0^h T(r, z, t) \sin(\lambda_m z) dz \tag{2.1}$$

where ξ_n is the nthpositive root of the transcendental equation

$$J_1(\xi_n a) = 0 \tag{2.2}$$

Taking the two transforms for the equation (1.1) and using conditions (1.3)-(1.5) we get

$$\frac{\partial \bar{\hat{T}}}{\partial t} + \mu_{mn}^2 \bar{\hat{T}} = k\lambda_m (-1)^{m+1} \hat{f}(\xi_n, t) \tag{2.3}$$

where

$$\lambda_m = \frac{m\pi}{h}; \quad \mu_{mn}^2 = k\left[\xi_n^2 + \lambda_m^2\right] \tag{2.4}$$

Solving (2.3) subject to condition (1.2) gives

$$\bar{\hat{T}}(\xi_n, \lambda_m, t) = k\lambda_m(-1)^{m+1} \int_0^t \hat{f}(\xi_n, t')e^{-\mu_{mn}^2(t-t')}dt' \tag{2.5}$$

Now using the inversion formulae for the finite Hankel transform and the Fourier sine transforms, we get

$$T(r, z, t) = \frac{4k}{a^2h} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{J_0(\xi_n r)\lambda_m(-1)^{m+1} \sin(\lambda_m z)A_{mn}(t)}{J_0^2(\xi_n a)} \tag{2.6}$$

where

$$A_{mn}(t) = \int_0^t \hat{f}(\xi_n, t')e^{-\mu_{mn}^2(t-t')}dt' \tag{2.7}$$

By carrying out an integration by parts in (2.7) can show that

$$A_m n(t) = \frac{1}{\mu_{mn}^2}[\hat{f}(\xi_n, t) - f(\xi_n, 0)e^{-\mu_{mn}^2 t} - \int_0^t [\frac{\partial}{\partial t'} f(\xi_n, t')]e^{-\mu_{mn}^2(t-t')}dt'] \tag{2.8}$$

where $\mu_{mn}$is given by (2.4) and $\xi_n$are the positive roots of the transcendental equation (2.2).

THE ALTERNATIVE SOLUTION

We introduce the so called pseudo-steady temperature distribution function $T_0(r, z, t)$ in which t is regarded as parameter and $T_0(r, z, t)$ is the temperature of the plate at time t. We now take $T_0(r, z, t)$ to satisfy the following equation

$$\frac{\partial^2 T_0}{\partial r^2} + \frac{1}{r}\frac{\partial T_0}{\partial r} + \frac{\partial^2 T_0}{\partial z^2} = 0 \tag{2.9}$$

subject to the boundary conditions

$$[T_0(r, z, t)]_{z=0} = 0 \tag{2.10}$$

$$[T_0(r, z, t)]_{z=h} = f(r, t) \tag{2.11}$$

$$\left[\frac{\partial T_0}{\partial r}\right]_{r=a} = 0 \tag{2.12}$$

Taking the finite Hankel transform and the finite Fourier transform of (2.9) and using (2.10)-(2.12) gives

$$\mu_{mn}^2 \bar{\hat{T}}_0 = k\lambda_m \hat{f}(\xi_n, t)(-1)^{m+1} \tag{2.13}$$

which using (2.3) may be written as

$$\frac{\partial}{\partial t}\left[\bar{\hat{T}} - \bar{\hat{T}}_0\right] + \mu_{mn}^2\left[\bar{\hat{T}} - \bar{\hat{T}}_0\right] = -\frac{\partial}{\partial t}\bar{\hat{T}}_0 \tag{2.14}$$

Solving the above equation and using (1.2) we find

$$\bar{\hat{T}}(\xi_n,\lambda_m, t) - \bar{\hat{T}}_0(\xi_n,\lambda_m, t) = -\bar{\hat{T}}_0(\xi_n, \lambda_m, 0)e^{-\mu_{mn}^2 t} - \int_0^t e^{-\mu_{mn}^2(t-t')}\frac{\partial}{\partial t'}\bar{\hat{T}}(\xi_n, \lambda_m, t')dt' \tag{2.15}$$

Substituting $\hat{\bar{T}}_0$ from (2.13) into (2.15) yields

$$\hat{\bar{T}}(\xi_n,\lambda_m,t) = \frac{k\lambda_m(-1)^{m+1}}{\mu_{mn}^2}\left[\hat{f}(\xi_n,t) - \hat{f}(\xi_n,0)e^{-\mu_{mn}^2 t} - \int_0^t e^{-\mu_{mn}^2(t-t')}\frac{\partial}{\partial t'}\hat{f}(\xi_n,t')dt'\right] \tag{2.16}$$

which, in view of (2.8) , becomes

$$\hat{\bar{T}}(\xi_n,\lambda_m,t) = k\lambda_m(-1)^{m+1}A_{mn}(t) \tag{2.17}$$

This equation is the same as the one given by (2.5) and (2.7).Therefore, inverting the two transforms we obtain $T(r,z,t)$ given by (2.6) and (2.8). A major difference between this solution and the one obtained in[3] is the positive sign with the integrand in (2.8). This makes the alternative solution in[3] erroneous.

3. DISPLACEMENT AND STRESS FUNCTIONS

The displacement function $\psi(r,z,t)$ is governed by equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = (1+\nu)a_t T \tag{3.1}$$

subject to the condition

$$\psi = 0 \text{ at } r = a \text{ for all time } t \tag{3.2}$$

where ν and a_t are Poisson's ratio and the coefficient of thermal expansion of the material of the plate respectively .The stress functions $\sigma_{rr}$and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu\frac{1}{r}\frac{\partial\psi}{\partial r}; \quad \sigma_{\theta\theta} = -2\mu\frac{\partial^2\psi}{\partial r^2} \tag{3.3}$$

where μ is Lame' constant , while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the plate in the plane state of stress. From (3.1) and (3.2) it follows that

$$\sigma_{\theta\theta} = -2\mu(1+\nu)a_t T - \sigma_{rr} \tag{3.4}$$

Thus $\sigma_{\theta\theta}$ is linear combination of T and σ_{rr}.

When the temperature T is of the form

$$T(r,z,t) = \sum_{n=1}^{\infty} E_n(z,t)J_o(\xi_n r) \tag{3.5}$$

it may be shown that

$$\psi(r,z,t) = -(1+\nu)a_t\sum_{n=1}^{\infty}\frac{E_n(z,t)}{\xi_n^2}\left[J_0(\xi_n r) - J_0(\xi_n a)\right] \tag{3.6}$$

Indeed, we substitute$T$ from (3.5) into (3.1), carry out the integration and make use of the condition (3.2) to arrive at (3.6). Substituting ψ from (3.6) into (3.3) yields

$$\sigma_{rr}(r,z,t) = -2\mu(1+\nu)a_t\sum_{n=1}^{\infty}E_n(z,t)\left[(r\xi_n)^{-1}J_1(\xi_n r)\right] \tag{3.7}$$

In the case of Problem 1, using (2.6) and (3.5) we see that

$$E_n(z,t) = \frac{4k}{a^2 h J_0^2(\xi_n a)} \sum_{m=1}^{\infty} \lambda_m (-1)^{m+1} \sin(\lambda_m z) A_{mn}(t) \tag{3.8}$$

where $A_{mn}(t)$ is given by (2.8) .Then ψ and $\sigma_{rr}$are given by (3.6)-(3.8).

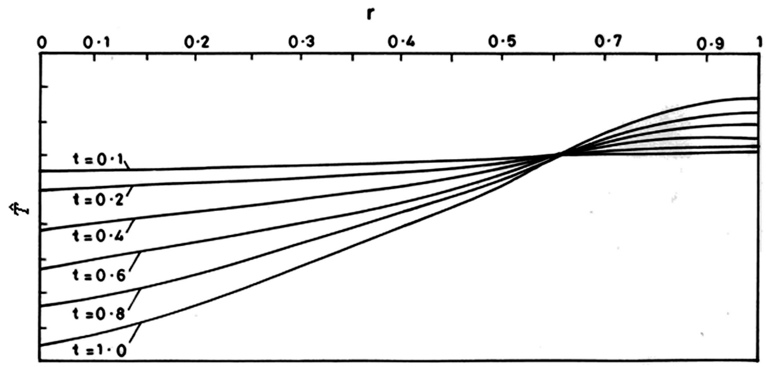

Figure 1

Variation of $\widehat{T} = T(r,z,t)/\alpha_1$ given by Eqns. (3.5), (4.6) and (4.4) with r for t = 0.1, 0.2,0.4,0.6,0.8,1.0. The values of other parameters are a =1, h =0.5, z = 0.3, k = 0.86. and $t_0 > 1$

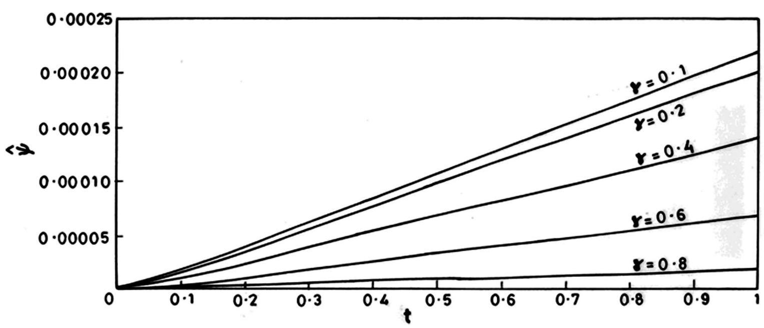

Figure 2

Variation of $\hat{\psi} = -\psi(r,z,t)\big/\beta_1$ given by equations (3.6), (4.6) and (4.4) with t for $r = 0.1, 0.2, 0.4, 0.6, 0.8$. The values of other parameters are the same as in Fig.1

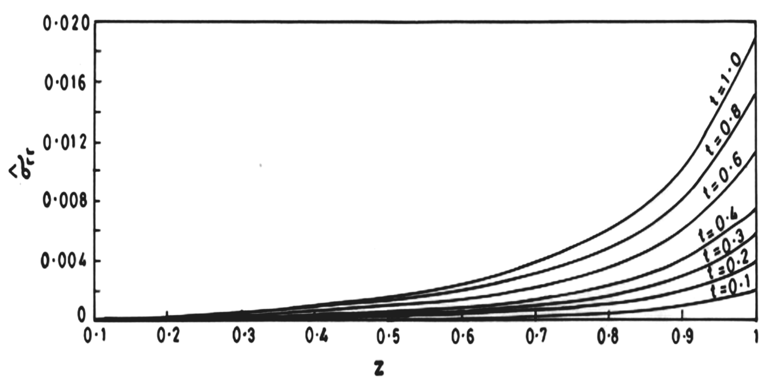

Figure 3

Variation of $\widehat{\sigma}_{rr} = -\sigma_{rr}(r,z,t)/\gamma_1$ given by Eqns.(3.7),(4.6) and (4.4) with z for t = 0.1,0.2,0.3,0.4,0.6,0.8,1.0.The values of other parameters are a =1, h = 1 , r = 0.0 , k = 0.86. and $t_0 > 1$

4. SPECIAL CASE OF PROBLEM 1

Let $f(r,t)$ in equation (1.4) be give by

$$f(r,t) = (ra - \frac{1}{2}r^2)\,[t - (t - t_0)H(t - t_0)] \tag{4.1}$$

where $H(x)$ is the Heaviside function. For brevity of discussion we restrict our attention to the case when $0 \leq t < t_0$ .Then we have

$$\hat{f}(\xi_{n,}t) = -aa_n t \big/ \xi_n^3 \tag{4.2}$$

$$a_n = \int_0^{x_n} J_0(y)dy;\; x_n = a\xi_n \tag{4.3}$$

Set

$$\alpha_1 = \frac{4\pi k}{a^2h^2};\; \beta_1 = -(1+\nu)a_t\alpha_1;\; \gamma_1 = 2\mu\beta_1 \tag{4.4}$$

From (2.7) and (4.2) we get

$$A_{mn}(t) = -\frac{aa_n}{\xi_n^3\mu_{mn}^2}\left[t + (e^{-\mu_{mn}^2 t} - 1)\big/\mu_{mn}^2\right] \tag{4.5}$$

which together with (3.8) and (4.4) yields

$$E_n(z,t) = -\frac{a\alpha_1 a_n}{\xi_n^3 J_0^2(\xi_n a)}\sum_{m=1}^{\infty}\frac{m(-1)^{m+1}\sin(\lambda_m z)}{\mu_{mn}^2} \times \left[t + (e^{-\mu_{mn}^2 t} - 1)\big/\mu_{mn}^2\right] \tag{4.6}$$

Thus the temperature $T(r,z,t)$ is then given by (3.5) and (4.6) .Also from(3.5)-(3.7) and (4.4) we note that the expression for $\psi(r,z,t)/\beta_1$can be obtained from $T(r,z,t)/\alpha_1$simply by replacing $J_0(\xi_n r)$by $\left[J_0(\xi_n r) - J_0(\xi_n a)\right]\big/\xi_n^2$. Similarly the expression for $\sigma_{rr}(r,z,t)/\gamma_1$ can be obtained from $T(r,z,t)/\alpha_1$ simply by replacing $J_0(\xi_n r)$ by $\left[(r\xi_n)^{-1}J_1(r\xi_n)\right]$.

The variation of $T(r,z,t)/\alpha_1$with $r$is shown in Fig.1 for $t = 0.1, 0.2, 0.4, 0.6, 0.8, 1.0$ by taking $a = 1, h = 0.5, z = 0.3, k = 0.86$ and $t_0 > 1$.The variation of $\psi(r,z,t)/\beta_1$ with t is shown in Fig.2 for $r = 0.1, 0.2, 0.4, 0.6, 0.8$ by taking $a = 1, h = 0.5, z = 0.3, k = 0.86$ and $t_0 > 1$. Finally, the variation of $\sigma_{rr}(r,z,t)/\gamma_1$ with z is shown in Fig. 3 for $t = 0.1, 0.2, 0.3, 0.4, 0.6, 0.8, 1.0$ by taking $a = 1, h = 1, r = 0.0$, $k = 0.86$ and $t_0 > 1$. For $r = 0$ the expression $[(r\xi_n)^{-1}J_1(r\xi_n)]$ is indeterminate and so the limit formula

$$\lim_{x\to 0}\left[x^{-1}J_1(x)\right] = \frac{1}{2} \tag{4.7}$$

is employed.

As $T(r,z,t)$ is given by (2.6) and (2.7). A more suitable form of $T(r,z,t)$is given by (3.5), (3.8) and (2.8).

REFERENCES

[1] Roy Choudhuri, S. K. (1973). A Note on Quasi-Static Thermal Deflection of a Thin Circular Plate due to Ramp Type Heating of a Concentric Circular Region of the Upper Face. *Journal of the Franklin Institute, 296*, 213-219.

[2] Meshram, S. A. and Deshmukh, K. C. (2003). Quasi Static Thermal Deflection of Thin Circular Plate. *Bull. Cal. Math. Soc., 95*, 101-106.

[3] Meshram, S. A., Deshmukh, K. C. (2004). Transient Heat Conduction in a Thin Circular Plate. *Bull. Cal. Math. Soc. 96*, 155-170.

[4] Buradkar, A. B. & Deshmukh, K. C. (2005). Theoretical Analysis of Thermal Stresses of a Thin Circular Plate. *Bull.Cal.Math.Soc. 97*, 297-310.

[5] Ölcer, N. Y. (1968). A General Class of Unsteady Heat Flow Problem in a Finite Composite Hollow Circular Cylinder. *Q.Appl. Math. 26*, 355-371.

[6] Sneddon, I. N. (1972). *The Use of Integral Transforms*. New York: McGraw Hill.